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LETTER TO THE EDITOR

Soliton solutions for Dirac equations with homogeneous non-linearity in (1 + 1) dimensions

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Abstract. Stationary state solutions with finite energy are presented for generic homogeneous non-linear Dirac equations in (1 + 1) dimensions. These models fall into two distinct classes, with dual existence conditions. In one case the soliton solution is unique and extends throughout the infinite line $\mathbb{R}^1$ while for the second class the soliton exists on a finite interval, and it is not unique.

Particular forms of non-linear Dirac equations can have localised finite-energy solutions. These solutions must necessarily be non-topological in the sense that, at large distance, the spinor field must approach the vacuum state $\psi = 0$. That follows from the fact that Dirac-like equations are linear in the time derivative and therefore the conserved charge, related to the global phase invariance of the spinor field, is merely a norm. Finiteness of this charge requires the field to vanish at infinity. (Clearly this comment also applies to non-linear Schrödinger equations.)

Furthermore the localised solutions must have a non-trivial time dependence. The only static solution is the vacuum state $\psi = 0$. Generally stationary state solutions are considered, for which the time dependence is $\exp(-i\omega t)$, with ω real.

In this work, solutions are obtained for generic Dirac equations with homogeneous non-linearities in (1 + 1) dimensions. Such models fall naturally into two distinct classes with distinct conditions required for the existence of localised finite-energy stationary states. These conditions are expressed in terms of the frequency and the non-linearity, when evaluated for a particular linear spinor field.

Consider the Lagrangian density

$$\mathscr{L} = i\bar{\psi}\gamma^\nu\partial_\nu\psi - \mu\bar{\psi}\psi - U(\bar{\psi}, \psi) \tag{1}$$

where ψ is a two-component spinor field, μ is a positive mass parameter and $U(\bar{\psi}, \psi)$ is a general homogeneous Lorentz invariant non-linearity with the property

$$\bar{\psi}\, \partial U(\bar{\psi}, \psi)/\partial\bar{\psi} = pU(\bar{\psi}, \psi) \tag{2}$$

with $p > 0$ but $\neq 1$. The representation used for the gamma matrices is

$$\gamma^0 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = i\sigma^1 = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

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The corresponding field equation for stationary states is

$$\sigma^1\, \mathrm{d}\psi/\mathrm{d}x = (\omega\sigma^3 - \mu - W(\bar{\psi}, \psi))\psi, \tag{4}$$

where

$$W(\bar{\psi}, \psi)\psi = \partial U(\bar{\psi}, \psi)/\partial\bar{\psi}. \tag{5}$$

Solutions of this equation must satisfy the identity

$$\omega\psi^+\psi - \mu\bar{\psi}\psi = U(\bar{\psi}, \psi). \tag{6}$$

This relation is obtained by multiplying (4) by $\mathrm{d}\bar{\psi}/\mathrm{d}x$, multiplying the Hermitian conjugate of (4) by $\sigma^3\, \mathrm{d}\psi/\mathrm{d}x$ (from the right), subtracting the results and integrating from x to infinity. Equations (4) and (6) lead to

$$\bar{\psi}\sigma^1\, d\psi/\mathrm{d}x = (1-p)U(\bar{\psi}, \psi). \tag{7}$$

Therefore the energy, associated with the Lagrangian density (1),

$$E = \int \bar{\psi}\sigma^1(\mathrm{d}\psi/\mathrm{d}x)\, \mathrm{d}x + \mu \int \bar{\psi}\psi\, \mathrm{d}x + \int U(\bar{\psi}, \psi)\, \mathrm{d}x, \tag{8}$$

can also be written in the form

$$E = \omega Q + (1-p) \int U(\bar{\psi}, \psi)\, \mathrm{d}x, \tag{9}$$

where Q is the conserved charge

$$Q = \int \psi^+\psi\, \mathrm{d}x. \tag{10}$$

It will now be shown that the solution of (4) with $U(\bar{\psi}, \psi)$ satisfying (2), is simply

$$\psi = (U(\bar{\chi}, \chi))^{1/(2-2p)}\chi, \tag{11}$$

where χ satisfies

$$\sigma^1\, \mathrm{d}\chi/\mathrm{d}x = (1-p)(\omega\sigma^3 - \mu)\chi. \tag{12}$$

Equation (12) is the usual Dirac equation with the spatial coordinate scaled by a factor $(1-p)^{-1}$. One first starts by multiplying (4) by $\bar{\psi}$ and then replaces the non-linearity by means of equation (6), with the result

$$\bar{\psi}[\sigma^1\, \mathrm{d}/\mathrm{d}x - (1-p)(\omega\sigma^3 - \mu)]\psi = 0. \tag{13}$$

It is easily verified that the most general solution (centred at the origin) of this equation is

$$\psi(x) = \phi(x)\chi_0, \tag{14}$$

where

$$\chi_0 = \begin{pmatrix} \cos \kappa(1-p)x \\ \xi \sin \kappa(1-p)x \end{pmatrix}, \tag{15}$$

$$\kappa = (\omega^2 - \mu^2)^{1/2}, \qquad \xi = [(\omega - \mu)/(\omega + \mu)]^{1/2}, \tag{16}$$

and $\phi(x)$ is determined by the substitution of (14) into (6), i.e.

$$\phi(x) = [U(\bar{\chi}_0, \chi_0)/(\omega - \mu)]^{1/(2-2p)}. \tag{17}$$

The factor $(\omega-\mu)$ can be absorbed in the definition of the linear spinor field χ_0: defining $\chi=\chi_0(\omega-\mu)^{-1/2}$ one reproduces the result (11).

The search for necessary and sufficient conditions for the existence of solitons is now reduced to the search of conditions which ensure the reality of $\phi(x)$ and the finiteness of the energy of $\psi(x)$. At this point one distinguishes two classes of homogeneous non-linearities:

(*a*) $p>1$. Under this condition, the non-linearity vanishes faster than the linear term in the field equation when $\psi\to 0$, that is

$$\lim_{\psi\to 0} W(\bar{\psi},\psi)=0. \tag{18}$$

This implies that the field equation is essentially linearised at large distance. A necessary condition for the field to approach zero at infinity is then $\omega^2<\mu^2$. A simple consequence of this inequality can be seen from (17): since $\omega-\mu<0$, $U(\bar{\chi}_0,\chi_0)$ must also be negative. This implies that $U(\bar{\psi},\psi)<0$ for ψ a non-trivial solution of (4) and therefore $(1-p)U(\bar{\psi},\psi)>0$. For a positive frequency solution, this automatically ensures the positivity of the energy density.

For these models, the linear spinor field can be written more conveniently in the form

$$\chi_0=\begin{pmatrix}\cosh\alpha(p-1)x\\ \beta\sinh\alpha(p-1)x\end{pmatrix}, \tag{19a}$$

$$\alpha=(\mu^2-\omega^2)^{1/2}, \qquad \beta=[(\mu-\omega)/(\mu+\omega)]^{1/2}. \tag{19b}$$

Known solutions, for the case $p=2$, (Lee *et al* 1975, Chang *et al* 1975, Kaus 1976) can now be easily reproduced. For the scalar case, $U=-\lambda(\bar{\psi}\psi)^2$, $\lambda>0$, the solution is given by

$$\psi=|\bar{\chi}\chi|^{-1}\chi, \qquad \chi=[\lambda/(\mu-\omega)]^{1/2}\chi_0. \tag{20}$$

Similarly, for a vectorial quartic non-linearity (where $(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)=(\psi^+\psi)^2$ since $\bar{\psi}\gamma^1\psi=0$ for $\psi(x)$ real) and a purely pseudoscalar quartic non-linearity (with $\gamma_5=\gamma^0\gamma^1$) the results are respectively $\psi=|\chi^+\chi|^{-1}\chi$ and $\psi=|\bar{\chi}\mathrm{i}\gamma_5\chi|^{-1}\chi$ with χ given by (20). It is easily verified that the charge associated with the solutions of the scalar and the vectorial non-linearity is finite, but it is infinite for the pseudoscalar case. Therefore there are no soliton solutions for a pseudoscalar quartic non-linearity (which contradicts the claim of Lee *et al* (1975)). This result is obviously true for any $p>1$. Actually it can be generalised to any pseudoscalar model satisfying the equality (18) (Mathieu and Morris 1984). The proof is based on the relation

$$\omega\int\bar{\psi}\psi\,\mathrm{d}x=\mu\int\psi^+\psi\,\mathrm{d}x+\mathrm{Re}\int\psi^+W(\bar{\psi},\psi)\psi\,\mathrm{d}x \tag{21}$$

which must be satisfied by any finite-energy solution of (4). For a pseudoscalar non-linearity, $\mathrm{Re}\,\psi^+W\psi=0$ and equation (21) implies that $\omega\int\bar{\psi}\psi\,\mathrm{d}x=\mu\int\psi^+\psi\,\mathrm{d}x$, or $|\omega|>\mu$, in contradiction with the existence condition $\omega^2<\mu^2$.

However, for homogeneous non-linearities, the non-existence of a soliton for the pseudoscalar case can be simply attributed to the fact that $U(\bar{\chi}_0,\chi_0)$ is not strictly negative for that case, being zero at the origin. A zero in $U(\bar{\chi}_0,\chi_0)$ leads to a singularity in the solution ψ and hence to an infinite energy.

In cases where solitons exist, it is simple to see that as $\omega\to 0$, the energy and the charge become infinite, illustrating the fact that static non-topological solitons do not exist.

Finally, if a soliton solution exists for a given model satisfying (18), it is unique in the sense that in the appropriate phase space there is only one trajectory ending at the point $\psi = 0$, this being the required asymptotic behaviour of finite energy solutions.

(b) $p < 1$. For models of this class,

$$\lim_{\psi \to 0} |W(\bar{\psi}, \psi)| = \infty. \tag{22}$$

The self interaction of the field produces an infinite effective potential well. A consistency condition for this self-trapping mechanism is that the well so produced must have the correct sign to support a bound state. For positive frequency solutions, this implies that at least for small values of the field, $U(\bar{\psi}, \psi) \geqslant 0$. However, since sgn $U(\bar{\chi}_0, \chi_0) = \text{sgn}\, U(\bar{\psi}, \psi)$, and that for a fixed frequency the sign of $U(\bar{\chi}_0, \chi_0)$ cannot change, to preserve the reality of $\phi(x)$ in (17), it follows that $U(\bar{\chi}_0, \chi_0)$ must be greater than or equal to 0 for all values of x which then forces the inequality $\omega > \mu$.

A typical model in this class is the fractional scalar non-linearity, where

$$U(\bar{\psi}, \psi) = b|\bar{\psi}\psi|^{p-1}\bar{\psi}\psi, \qquad b > 0, \tag{23}$$

(Mathieu and Saly 1984, Mathieu 1985, Mathieu and Morris 1985). The positivity of $U(\bar{\psi}\psi)$ for the solutions is now translated into the requirement $\bar{\psi}\psi \geqslant 0$. This condition then implies that the solution (11) is valid only for values of x such that

$$\cos^2(1-p)\kappa x \geqslant \xi^2 \sin^2(1-p)\kappa x \tag{24a}$$

or

$$|x| \leqslant X = [(1-p)\kappa]^{-1} \tan^{-1} \xi^{-1}. \tag{24b}$$

For $|x| > X$, the solution is continued by $\psi = 0$. Hence the support of the soliton is compact, of total length $2X$.

In contradistinction with models of the first class, a pseudoscalar non-linearity satisfying (22) leads to a field equation whose solutions are solitons with compact support. (This is particularly interesting in relation to the fact that the model (23) furnishes a natural field theoretical generalisation of the MIT bag model (Chodos *et al* 1974) in the limit $p \to 0$ (Mathieu 1985). Hence the existence of solitons for the pseudoscalar case ensures that a chiral extension of the model is possible.)

On the other hand, a rather surprising result is that for a fractional vectorial non-linearity, $U = b(\psi^+\psi)^p$, no finite energy solutions exist because nowhere does the charge density vanish, that is $U(\bar{\chi}_0, \chi_0) > 0$ everywhere. This shows that a necessary condition for the existence of a soliton is that $U(\bar{\chi}_0, \chi_0) = 0$ at some points (and if it is true for two values of x, $x = \pm X$, it is true for an infinite number of values of x since $U(\bar{\chi}_0, \chi_0)$ is a periodic function). The condition that $U(\bar{\chi}_0, \chi_0)$ vanishes at some points also ensures the compactness of the soliton: the support is then the length of the interval over which $U(\bar{\chi}_0, \chi_0) > 0$; outside of this interval, the solution is continued by the vacuum.

Note that for non-homogeneous non-linearities which satisfy (22), the existence of the soliton does not ensure the compactness of their support. The simplest counter-example is the logarithmic model

$$U = -\lambda\bar{\psi}\psi \ln|\bar{\psi}\psi|/\beta^2, \qquad \lambda > 0, \tag{25}$$

where β^2 is a parameter, for which

$$\lim_{\psi\to 0}(W(\bar{\psi}, \psi) = -\lambda - \lambda \ln|\bar{\psi}\psi|/\beta^2) = \infty. \tag{26}$$

The solution is easily found to be

$$\psi(x) = |\beta|\exp\left[-\left(\frac{\omega-\mu}{2\lambda}\right)(\cosh^2 \lambda x + \xi^{-2}\sinh^2 \lambda x)\right]\begin{pmatrix}\cosh \lambda x\\ \sinh \lambda x\end{pmatrix}. \tag{27}$$

For $\omega > \mu$, it describes a smooth lump with $\bar{\psi}\psi > 0$ and $\psi \to 0$ as $x \to \pm\infty$; thus the support of the soliton is not compact.

For compact solitons, there is a weak superposition principle—the sum of two solutions is also a solution as long as they do not overlap. As a result the fundamental soliton (11) is not unique.

Finally, the condition $\omega^2 > \mu^2$ automatically rules out static solutions. When $\omega = \mu$, the solution is the boxed plane wave which has infinite energy. Hence there is no plane wave sector for these models: all the finite energy solutions are localised.

Summarising the results, one finds that existence conditions for positive frequency stationary state soliton solutions for $p > 1$ are $\omega < \mu$ and $U(\bar{\chi}_0, \chi_0) < 0$ (and nowhere zero) while for $p < 1$, $\omega > \mu$ and $U(\bar{\chi}_0, \chi_0) > 0$ over a finite interval and zero at the end of the interval. There is then an apparent duality between the existence conditions of these two distinct classes. By construction, these conditions are necessary and sufficient. In the first case, the soliton is unique (for a given ω and fixed values of the parameter) and infinitely extended; the reverse applies for the solitons in models of the second class.

For both cases, it has been shown that $(1-p)U(\bar{\psi}, \psi) \geq 0$ is required, where ψ is a solution. It has already been shown that, under a reasonable continuity hypothesis, the integral form of this condition

$$(1-p)\int U(\bar{\psi}, \psi)\, \mathrm{d}^n x = \int (U(\bar{\psi}, \psi) - \bar{\psi}W(\bar{\psi}, \psi)\psi)\, \mathrm{d}^n x > 0 \tag{28}$$

is a necessary condition for the existence of localised solutions for models satisfying (18) (Mathieu and Morris 1984). (The analysis presented there applies in any number of spatial dimensions.) The present analysis illustrates this result and hints that the local version of (28) could actually be necessary in higher spatial dimensions for arbitrary models.

As emphasised by Kaus (1976), one motivation for the study of non-linear Dirac equations in $(1+1)$ dimensions is that these equations correspond to the asymptotic form of the equations in the physically interesting case of $(3+1)$ dimensions. Hence some qualitative properties of the solitons could be similar in the two cases.

A final comment, of a more general character, will conclude this work. It has been pointed out that charge conservation for the Dirac equation prevents the existence of topological solitons. Finiteness of the charge also precludes static non-topological solitons. Positive energy static solitons are also excluded in non-linear scalar field theories (non-linear Klein-Gordon equations) with no degenerate vacuum. This suggests that models with no topological solitons cannot have stable static localised solutions. Rajaraman has found a static non-topological soliton in a particular two-component scalar field theory (Rajaraman 1979). In his model there are also two classes of topological solitons. A remarkable energy sum rule, which relates the energy of these three types of solitons (Subbaswamy and Trullinger 1980, 1981) supports the fact that static topologically trivial solitons could only occur as a combination of existing topological states. This is also illustrated by the work of Mukherjee (1985) where the construction of static non-topological solitons from topological ones is explicit.

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